

TOROIDAL MAPS WITH PRESCRIBED TYPES OF VERTICES AND FACES

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1. *Introduction.* Let a 3-connected graph G be embedded in an orientable surface of genus g in such a way that the connected components of its complement in the surface are topological discs. We denote by $v_k(G)$ the number of k -valent vertices of G and by $p_k(G)$ the number of k -gonal faces of the map defined by the embedded graph. For the numbers $v_k(G)$ and $p_k(G)$, it follows from Euler's formula that

$$\sum_{k \geq 3} (4 - k) p_k(G) + \sum_{k \geq 3} (4 - k) v_k(G) = 8(1 - g). \quad (1)$$

Since $\sum_{k \geq 3} k p_k(G)$ and $\sum_{k \geq 3} k v_k(G)$ equal twice the number of edges of G , both these numbers are even and so of course are the numbers $\sum_{k \geq 3} k p_k(G)$ and $\sum_{\substack{k \geq 3 \\ k \neq 4}} k v_k(G)$. However, the converse of (1) and of the statement just mentioned does not hold. We are as yet far from knowing the answer to the general question: Given sequences $p = (p_3, p_5, \dots, p_m)$ and $v = (v_3, v_5, \dots, v_n)$ satisfying

$$\sum_{k \geq 3} (4 - k) p_k + \sum_{k \geq 3} (4 - k) v_k = 8(1 - g), \quad (2)$$

and

$$\sum_{k \geq 3} k p_k, \sum_{k \geq 3} k v_k \text{ are even,} \quad (3)$$

for what numbers p_4, v_4 does there exist a graph G embedded in a surface of genus g for which $p_k(G) = p_k$ and $v_k(G) = v_k$, ($k \geq 3$) holds?

Only the following result was published (Grünbaum [2]; for references on this and related problems, see Grünbaum [1; 3], Grünbaum–Shephard [4]). Given sequences $p = \{p_i\}$ and $v = \{v_i\}$, satisfying (2) and (3) with $g = 0$, there exists a graph G embedded in the sphere such that $p_i(G) = p_i$ and $v_i(G) = v_i$ ($i \geq 3$, $i \neq 4$). (Such sequences are said to be realizable on the sphere.) The aim of the present paper is to prove an analogous statement for $g = 1$ (the torus). Although some ideas of Grünbaum [2] are employed the procedure of our proof differs from his. Our result is contained in the following

THEOREM. *For any pair of sequences $p = (p_3, p_5, \dots)$ and $v = (v_3, v_5, \dots)$, satisfying (2) and (3) with $g = 1$, a map M on the torus realizing them exists, if, and only if, the pair p and v is different from $p = (1, 1, 0, 0, \dots)$ and $v = (0, 0, \dots)$ or $p = (0, 0, \dots)$ and $v = (1, 1, 0, 0, \dots)$.*

2. *Proof of the Theorem.* First we prove the non-existence of a map on the torus with a regular quadrivalent graph having one triangle (and one pentagon) only.

Let G be a 4-valent graph embedded on the torus. An edge bc is a direct extension of an edge ab if the other two edges meeting b lie on opposite sides of the path abc .

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A path $A_1 A_2 A_3 \dots A_n$ is a geodesic arc if the edge $A_i A_{i+1}$ is a direct extension of the edge $A_{i-1} A_i$; it is a closed geodesic if $A_n = A_1$, and $A_n A_2$ is a direct extension of $A_{n-1} A_1$.

Suppose $p_3(G) = p_5(G) = 1$, $p_i(G) = 0$ for $i > 5$, $v_i(G) = 0$ for $i \neq 4$, for the map on the torus defined by the graph G . Take a simple closed curve Γ in G which is a geodesic arc. From Grünbaum [1], p. 239–241, it follows that, if Γ bounds a topological disc D , then there are at least two triangles of G in D .

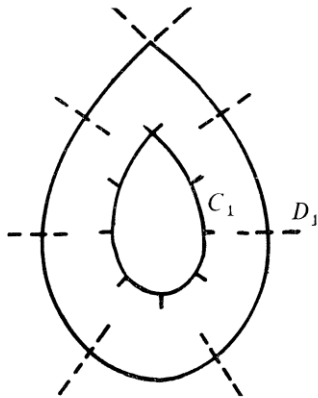


Fig. 1

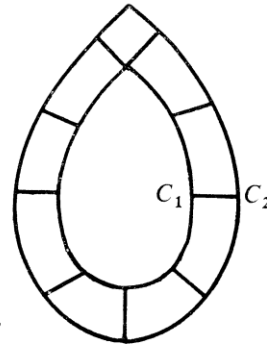


Fig. 2

Let the geodesic arc Γ not bound a topological disc. We can cut the torus apart along Γ and obtain an annulus A . Let the inner boundary (see Fig. 1) of the annulus be C_1 and the outer boundary be D_1 . The number of cases which must now be considered is large. We shall treat only some of them, the remainder being left to the reader.

Case 1. C_1 is not a closed geodesic, i.e. one vertex of C_1 meets three faces in A , and one vertex of D_1 meets only one face in A .

We consider the faces in the annulus meeting C_1 . If they are all 4-sided then these 4-sided faces meet another closed curve that is a geodesic, which we shall call C_2 . (See Fig. 2.)

If all faces that meet C_2 and are between C_2 and D_1 are 4-sided then these faces determine a third geodesic C_3 . We continue this argument until we reach a geodesic C_n that meets either a pentagon or a triangle. We now do the same with faces meeting

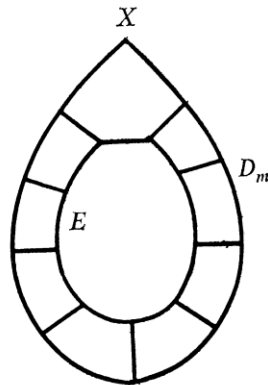


Fig. 3

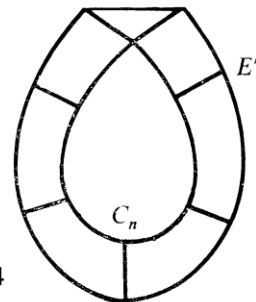


Fig. 4

D_1 , finding successive geodesics D_i until one of them, D_m , meets a pentagon or a triangle.

Case 1a. A pentagon meets that vertex X in the annulus A' , bounded by D_m and C_n , which meets only one face in A' . All other faces meeting D_m are 4-sided. (See Fig. 3.)

These faces will determine a closed geodesic E in A' . We now turn to C_n which must meet the triangular face.

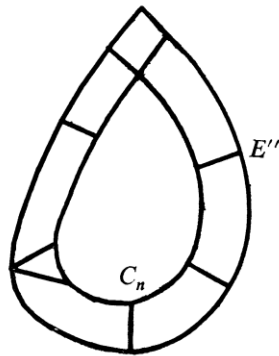


Fig. 5

Case 1a₁. The triangle meets C_n as in Fig. 4. All other faces meeting C_n will be 4-sided. These faces will determine a closed geodesic E' between C_n and D_m . All other faces of G are 4-sided and form concentric "rings" around E' . We now reach a contradiction because E has fewer vertices than C_1 , and when we add concentric rings of 4-sided faces around E' this number of vertices remains the same thus E and E' have the same number of vertices. This, however, implies that D_1 has more vertices than C_1 .

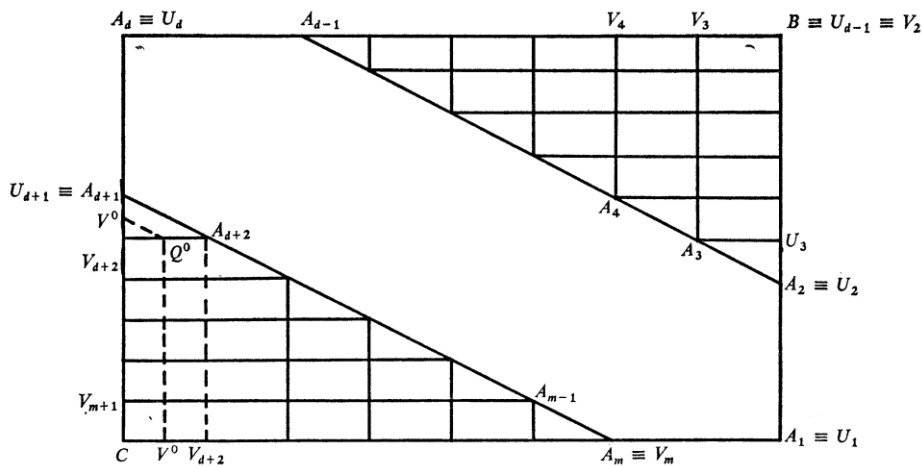


Fig. 6

Case 1a₂. The triangle has an edge on C_n. (Fig. 5). All other faces meeting C_n will be 4-sided, thus these faces will determine a closed curve, E'', composed of two geodesic arcs. The other 4-sided faces again form concentric "rings" around E''. The outer boundaries of each of these rings will be closed curves consisting of two geodesic arcs. This means that E consists of two geodesic arcs instead of one which is a contradiction.

There are several other cases, all depending on the placement of the pentagon and triangle. The proof becomes tedious and requires repeated applications of the kinds of arguments in cases 1a₁ and 1a₂.

Case 2. C₁ is a closed geodesic. This case reduces to Case 1 as soon as a triangle or a pentagon occurs in the annulus.

The non-realizability on the torus of the sequences p = (0, 0, ...), v = (1, 1, 0, 0, ...) follows now by duality from the considerations above.

To prove the sufficiency of the conditions mentioned in our Theorem we shall construct for each pair of sequences p = (p₃, p₅, ..., p_m), v = (v₃, v₅, ..., v_n) a toroidal map realizing them.

I. Let

$$\sum_{k \geq 3} (4 - k)p_k = \sum_{k \geq 3} (4 - k)v_k = 0.$$

(a) v_i = 0 for all i ≠ 4. For $\sum_{k \geq 3} p_k = 0$ the surface of a polyhedron of genus 1 which we get by joining k ≥ 3 cut-off n-gonal prisms together is an example of a desired cell decomposition of the torus. From now on we may assume that $\sum_{k \geq 3} p_k > 1$.

We begin the construction with an m-gon M, m = 4 + $\sum_{k \geq 5} (k - 4)p_k$, with vertices A₁, A₂, ..., A_m with right angles in the vertices A₁ and A_d, d = [½m] + 1

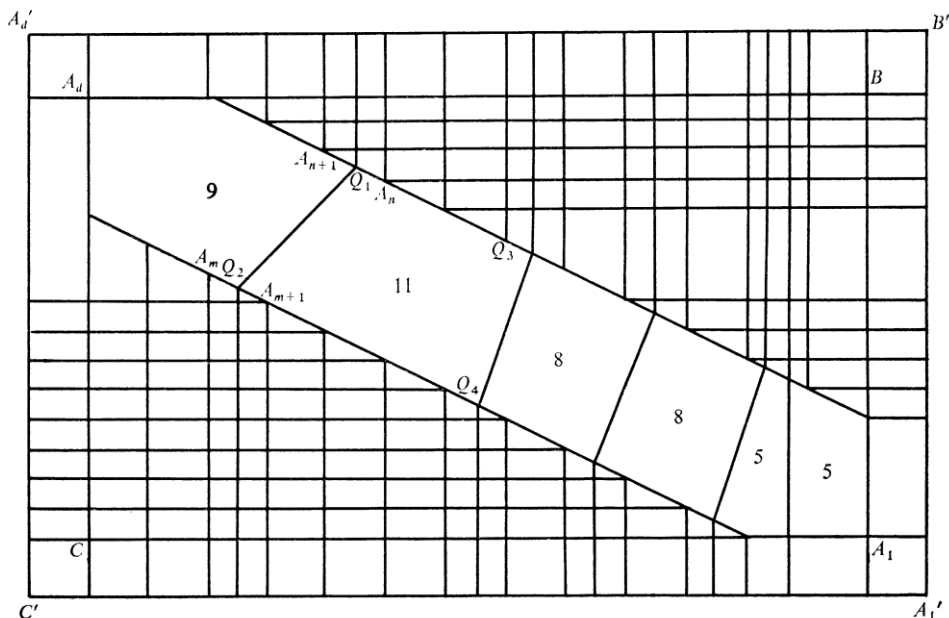


Fig. 7

(See Fig. 6). Let B be the point of intersection of the lines $A_1 A_2$ and $A_d A_{d-1}$, and C that of $A_m A_1$ and $A_d A_{d+1}$. The rays from the points A_i in the direction away from M and parallel to the line $A_1 C$ intersect the segment $A_1 B$ or $A_d C$ in the points U_i . Analogously we obtain the points V_i on the segments $A_1 C$, BA_d . Essential for the construction is the fact that if $\sum_{k \geq 5} k p_k$ is even then the segments $A_1 B$, CA_d or CA_1 , $A_d B$ contain the same number $d-1$ of vertices U_i or V_i . If $\sum_{k \geq 5} k p_k$ is odd then the segments $A_1 B$ and BA_d contain $d-1$ points U_i or V_i , but in each of the segments $A_d C$, CA_1 there are only $d-2$ vertices V_i or U_i . In this case we add a new edge $U^0 Q^0$ and a series of edges on the arc $Q^0 V^0$ where the point U^0 , Q^0 or V^0 lies between the point U_{d+1} and U_{d+2} , U_{d+2} and A_{d+2} or C and V_{d+2} respectively (dashed lines on Fig. 6). This operation will be called "balancing the number of vertices". Thus all sides of the rectangle $A_1 B A_d C$ have the same number of points U_i or V_i .

Next we cut off from the polygon M the needed k -gons. Let k be such that $p_k \neq 0$. We choose a point Q_1 between A_n and A_{n+1} , $n < d$, and a point Q_2 between A_m and A_{m+1} , $m > d$, such that $Q_1 A_{n+1} A_{n+2} \dots A_d A_{d+1} \dots A_m Q_2$ is a k -gon. The point Q_1 is joined with a point between V_n and V_{n+1} and the point Q_2 with a point between V_m and V_{m+1} by an arc. Analogously we choose points Q_3 or Q_4 between A_v and A_{v+1} or A_u and A_{u+1} so that $Q_3 A_{v+1} A_{v+2} \dots Q_1 Q_2 A_{m+1} \dots Q_4$ is a t -gon for such a t for which $p_t \neq 0$ etc. (See Fig. 7). This can always be done so as to dissect M into the desired k -gons. A little caution must be exercised in choosing the points A_n , A_m , A_u , A_v in order to balance up the numbers of edges of the halves of M used. The graph we get has again the property that on opposite sides of the rectangle $A_1 B A_d C$ there are equal numbers of vertices. Along all sides of this rectangle we add a series of quadrangles. The map we get is called the p -component of the map to be constructed.

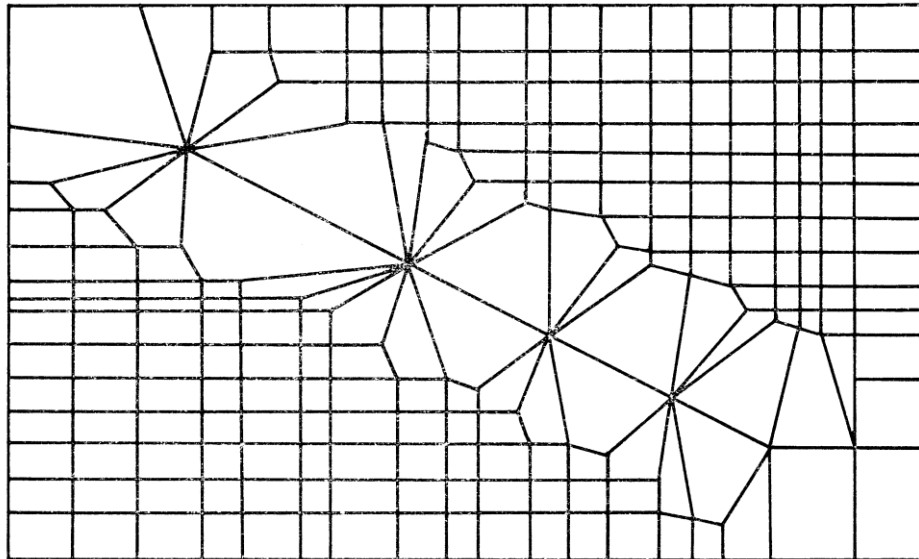


Fig. 8

The last stage of the construction is obvious. Vertices on opposite sides of the rectangle $A_1' B' A_d' C'$ are identified pairwise. First a tube is obtained and then the vertices on its borders are identified.

(b) For some $i \neq 4$, $v_i \neq 0$, and for some $j \neq 4$, $p_j \neq 0$, but none of the given sequences is of the form $(1, 1, 0, 0, \dots)$. First the p -component is constructed as in (a). In the same way the dual v^* of the v -component of the map is constructed. It consists of $p_i^* = v_i$ i -gonal faces for all $i > 4$, and quadrangles and triangles. By dualizing this map and by modifying the "border" of that dual map we get the v -component of our map; in Fig. 8 the dual of the map in Fig. 7 is shown. Both the p -component

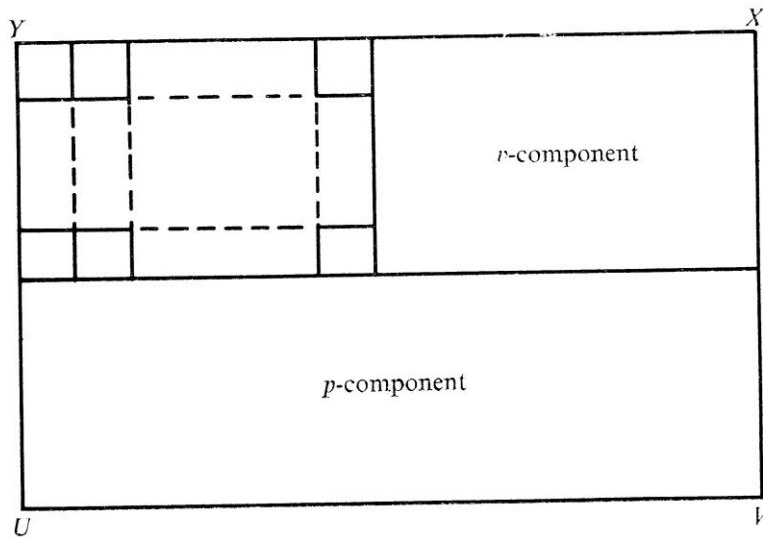


Fig. 9

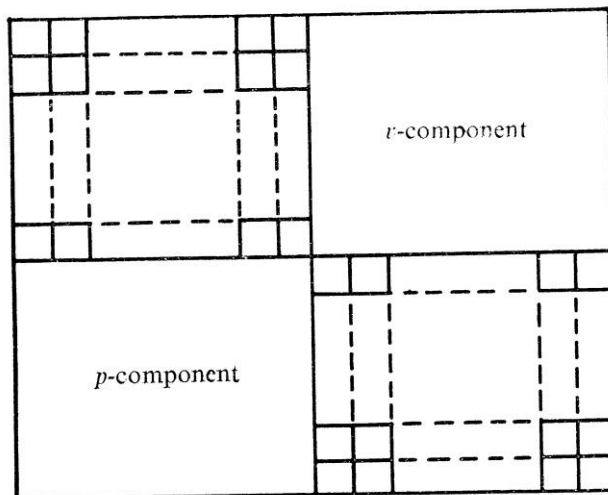


Fig. 10

and v -component are joined and, if needed, supplemented by quadrangles (See Fig. 9) so as to be able to perform the concluding stage of the construction as in case (a).

(c) *The case $p_j = 0$ for all $j \neq 4$ is dual to (a).* First the dual v^* of the v -component is constructed.

(d) *None of the sequences p, v is of the form $(0, 0, \dots)$, but at least one of them is of the form $(1, 1, 0, 0, \dots)$.* The sequence $(1, 1, 0, \dots, 0)$ cannot be realized as a p - or v -component with equal numbers of vertices on opposite sides of the rectangle. If $p = (1, 1, 0, \dots, 0)$ and $v = (1, 1, 0, \dots, 0)$ the p -component and v -component can be joined “diagonally” in such a way to get, after supplementing by quadrangles, equal numbers of vertices on opposite sides of the rectangle $UVXY$. (See Fig. 10).

If one of the sequences, *i.e.*, v , is of the form $(1, 1, 0, \dots, 0)$ then in the p -component the operation “balancing the number of vertices” is performed and the joining of the p - and v -components is performed as before.

II. *Let*

$$\sum_{k \geq 3} (4 - k)p_k > \sum_{k \geq 3} (4 - k)v_k.$$

First we construct the p -component and the v -component realizing the sequences $p' = (p_3', p_5', \dots, p_m')$ and $v' = (v_3', v_5', \dots, v_n')$ such that $p_i' = p_i$ and $v_i' = v_i$ for all $i \geq 5$, and $v_3' = \sum_{k \geq 5} (k - 4)v_k$ and $p_3' = p_3 - (v_3' - v_3)$ holds.

Next $v_3' - v_3$ trivalent vertices in the v -component must be changed into triangles. Notice that $v_3' - v_3 = d$ is always even because $\sum_{k \geq 3} kv_k$ as well as $\sum_{k \geq 3} kv_k'$ is even; therefore we can construct rays from $\frac{1}{2}d$ trivalent vertices going to one border of the v -component changing these vertices into 4-valent ones and forming one triangle (and possibly quadrangles) each. The same can be done with $\frac{1}{2}d$ trivalent vertices lying “on the other side” of the v -component so as to get equal numbers of vertices on opposite borders of the v -component. (See Fig. 11, where 4 trivalent vertices are changed into triangles.) The concluding stage of the construction, *i.e.* forming the tube etc., can now be performed as in part I.

III. *The case*

$$\sum_{k \geq 3} (4 - k)p_k < \sum_{k \geq 3} (4 - k)v_k,$$

is settled by dualization from the case II. The proof of the Theorem is finished.

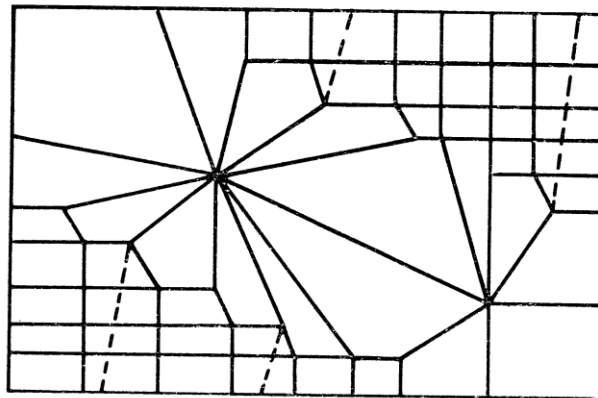


Fig. 11

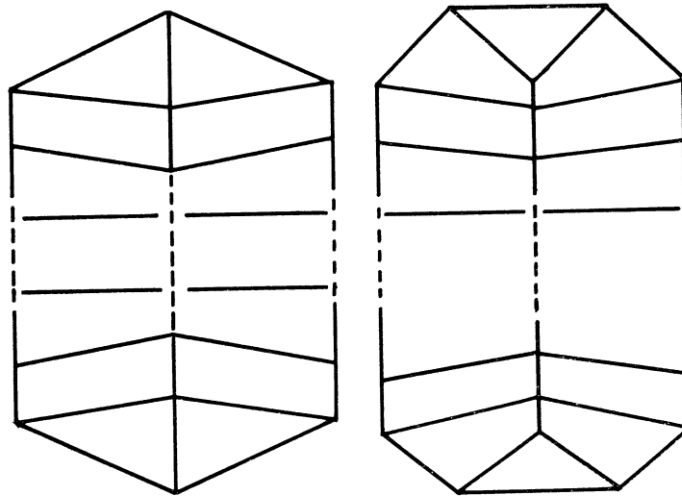


Fig. 12

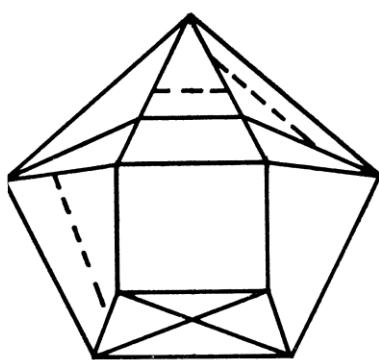


Fig. 13

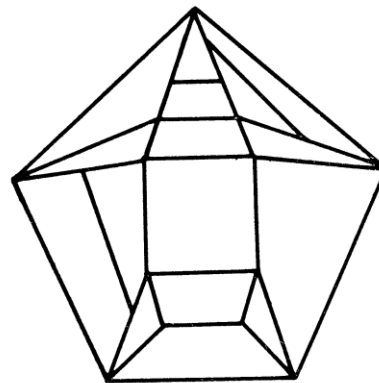


Fig. 14

Remark 1. The construction used in the proof of our Theorem is applicable, with few changes, for maps on the sphere too. As above we construct the map on the “tube” realizing sequences $p' = \{p'_i\}$, $v' = \{v'_i\}$ such that $p'_3 \leq p_3$, $v'_3 \leq v_3$, $p'_i = p_i$, $v'_i = v_i$ for all $i \geq 5$ holds, and $\sum_{i \geq 3} (4 - i)(p'_i + v'_i) = 0$ and $\sum_{k \geq 3} kp'_k$, $\sum_{k \geq 3} kv'_k$ are even. The openings of the tube are closed by one of the maps on Fig. 12 or their duals. In general for the map M , which we obtain by such a construction, the number $p_4(M) + v_4(M)$ is smaller than that obtained by the construction in Grünbaum [2]. There are some sequences for which this procedure does not work. The graphs of these maps are those in Fig. 13 (the dashed edges are added successively to increase the number of trivalent vertices) and Fig. 14, or their duals.

Remark 2. The non-existence of the decomposition of an annulus A treated in the first part of the proof of the Theorem was proved independently by W. Meyer (letter dated May 1970). J. Zaks has found a construction, different from ours,

proving the existence of a quadrivalent toroidal map realizing any sequences p, v satisfying (2) and (3) with $v_i = 0$ ($i \neq 4$), different from $p = (1, p_4, 1, 0 \dots)$ (letter dated February 1970).

Remark 3. If we are given sequences $\{p_i\}, \{v_i\}$ with all p_i and v_i even satisfying (2) and (3) with $g = 0$, we can use the following construction to obtain a centrally symmetric 3-polytope realizing $\{p_i\}, \{v_i\}$. Let $p_3' \leq p_3, v_3' \leq v_3$ be integers such that $p_3' + v_3' = 8$ and $p_3', v_3' \equiv 0 \pmod{4}$ holds. On a face F of a cube draw the p - and v -components realizing $p' = (\frac{1}{2}(p_3 - p_3'), \frac{1}{2}p_5, \dots), v' = (\frac{1}{2}(v_3 - v_3'), \frac{1}{2}v_5, \dots)$. Next we project this configuration through the centre of the cube onto the opposite face. On a face adjacent to F draw that one from among the graphs on Fig. 12 or their duals which has p_3' triangles and v_3' trivalent vertices. Then project this graph in the same manner through the centre of the cube. Further, we add edges across the remaining two faces to get only quadrangles on them, and 4-valent vertices on the edges of the original cube. To get a centrally symmetric polytope with this configuration of faces we use Grünbaum's theorem which characterizes the graphs of centrally symmetric polytopes [1; p. 245].

It appears that this construction will work unless $v_5 + p_5 = 2$ and $v_3 + p_3 = 10$ holds. In fact in these cases we are forced to place on the face F of the cube one pentagon (or a 5-valent vertex) and one triangle (or a trivalent vertex) which cannot be done so as to get equal numbers of vertices on opposite borders of F . In the other cases these difficulties do not occur.

The above argument gives an almost complete answer to Grünbaum's [1; p. 269] conjecture concerning the realizability of sequences by 4-valent centrally symmetric 3-polytopes.

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